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Quantal canonical symmetry for a constrained Hamiltonian system

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Abstract. Starting from the phase-space generating functional of the Green's function for a constrained Hamiltonian system, the canonical Ward identities under the global symmetry transformation in phase space is deduced. The local transformation connected with this global symmetry transformation is studied, the conserved charges are obtained at quantum level if the effective canonical action is symmetric (the constraints are also invariant under the transformation) and, therefore, the canonical Noether theorem in the quantum case is obtained. The generalized canonical Ward identities under the local transformation has been deduced. We give a preliminary application to a system of interacting polaron with photon. The conserved charges and Ward identities for proper vertices are obtained, but we do not carry out the integration over the canonical momenta in the phase-space generating functional as usually performed.

1. Introduction

The connection between continuous symmetry and conservation laws is usually referred to as Noether's theorem in the classical theories. All of these discussions are based on the examination of the Lagrangian in configuration space and the corresponding transformation expressed in terms of Lagrange's variables. The system with a singular Lagrangian is subject to some inherent phase space constraint which is, in fact, a constrained Hamiltonian system (Sundermeyer 1982). The generalization of Noether's theorem to a system with singular Lagrangian in terms of canonical variables was discussed by Li and Li (1991) and Li (1991).

Ward identities (or Ward–Takahashi identities) and their generalization play an important role in modern quantum field theories (Ward 1950, Takahashi 1957, Slavnov 1972, Taylor 1971). They are useful tools for renormalization of field theories and calculations in practical problems (for example, in QCD). Ward identities have been generalized to the supersymmetry (Joglekar 1991) and superstring theories (Danilov 1991) and other problems. All these derivations for Ward identities in the functional integration method are usually discussed by using a configuration-space generating functional (Surra and Young 1973, Lhallabi 1989) which is valid for the case when the integration over canonical momenta belongs to the Gauss- or Feynman-type category. Phase-space path integrals are more basic than configuration-space path integrals; the latter provide for a Hamiltonian quadratic in the canonical momenta, whereas the former apply to arbitrary Hamiltonians. Thus, phase-space form of the path integral is a necessary precursor to the configuration-space form (Mizrahi 1978). While the phase-space generating functional cannot be simplified by carrying out explicit integration over the canonical momenta, even if the integration over the momenta can be carried out, the effective Lagrangian sometimes shows singularity with a δ -function (Lee and Yang 1962, Gerstein *et al* 1971, Du *et al* 1980). Especially for the constrained Hamiltonian system with complicated constraints, it is very difficult or even impossible to carry out the integration over the canonical momenta (Nishikawa 1993). The properties of the generating functional in phase space under local transformation of the canonical variables was discussed and the canonical Ward identities in phase space deduced by using these transformation properties in Li (1994a–c).

In this paper the canonical quantal symmetry for a constrained Hamiltonian system will be further studied. Based on the phase-space generating functional of Green's function for a system with a singular Lagrangian, the generalized Ward identities under global and local transformation in phase space are deduced. The local transformation corresponding to the global symmetry transformation is considered and the conserved charge is obtained at the quantum level if the effective canonical action is symmetric and the constraint conditions are invariant under those transformations. Thus, the canonical Noether's theorem in the quantum case is also deduced. We have generalized the canonical Ward identities to a more general case for the local transformation. We give a preliminary application to a system containing phonons, electrons and photons which can be described in terms of a singular Lagrangian; the conserved charges and generalized Ward identities for proper vertices are deduced.

2. Global symmetry and canonical Ward identities

Consider a system described by the field variables $\varphi^\alpha(x)$ ($\alpha = 1, 2, \ldots, n$), where α denotes an index for different fields or different components of a field. The Lagrangian of the field is $\mathcal{L}(\varphi^\alpha, \varphi^\alpha_{,\mu})$ where $\varphi^\alpha_{,\mu} = \partial_\mu\varphi^\alpha$, $\partial_\mu = \partial/\partial x^\mu$ and $x^\mu = (t, \vec{x}\,)$. For a system with a singular Lagrangian whose extended Hessian matrix is degenerate (Sundermeyer 1982).

Due to the singularity of the Lagrangian, the motion of this system is restricted to a hypersurface of the phase space, determined by a set of constraints. Let Λ_k ($k = 1, 2, \ldots, K$) be the first-class constraints and θ_i ($i = 1, 2, \ldots, I$) be the second-class constraints. The gauge conditions associated with the first-class constraints are Ω_k ($k = 1, 2, \ldots, K$). The phase-space generating functional of Green's function for this constrained Hamiltonian system is given by (Li 1994a)

$$Z[J, K] = \int \mathcal{D}\varphi^\alpha \mathcal{D}\pi_\alpha \prod_{i,k,l} \delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l) \det|\{\Lambda_k, \Omega_l\}|[\det|\{\theta_i, \theta_j\}|]^{1/2} \times \exp\left\{ i\left[I^{\mathrm{p}} + \int \mathrm{d}^4x\,(J_\alpha\varphi^\alpha + K^\alpha\pi_\alpha)\right]\right\}. \tag{1a}$$

Using the δ-function and integral properties of the Grassmann variables $C_l(x)$ and $\overline{C}_k(x)$, one obtains

$$Z[J, K] = \int \mathcal{D}\varphi^\alpha \mathcal{D}\pi_\alpha \mathcal{D}\lambda_m \mathcal{D}C_l \mathcal{D}\overline{C}_k \exp\left\{ i\left[I^{\mathrm{p}}_{\mathrm{eff}} + \int \mathrm{d}^4x\,(J_\alpha\varphi^\alpha + K^\alpha\pi_\alpha)\right]\right\} \tag{1b}$$

where

$$I^{\mathrm{p}}_{\mathrm{eff}} = \int \mathrm{d}^4x\, \mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} = \int \mathrm{d}^4x \left\{ \mathcal{L}^{\mathrm{p}} + \lambda_i\theta_i + \lambda_k\Lambda_k + \lambda_l\Omega_l \right.$$

$$+\int \mathrm{d}^4y\,[\overline{C}_k(x)\{\Lambda_k(x),\Omega_l(y)\}C_l(y)+\tfrac{1}{2}\overline{C}_i(x)\{\theta_i(x),\theta_j(y)\}C_j(y)]\Bigg\} \tag{2}$$

and the canonical action is

$$I^{\mathrm{p}}=\int \mathrm{d}^4x\,\mathcal{L}^{\mathrm{p}}=\int \mathrm{d}^4x\,(\pi_\alpha\dot{\varphi}^\alpha-\mathcal{H}_{\mathrm{c}}) \tag{3}$$

where the π_α are canonical momenta conjugate to φ^α, $\pi_\alpha=\partial\mathcal{L}/\partial\dot{\varphi}^\alpha$, $\mathcal{H}_{\mathrm{c}}$ is a canonical Hamiltonian of the system, $\lambda_i(x)$, $\lambda_k(x)$ and $\lambda_l(x)$ are multiplier fields, $\lambda_m=(\lambda_i,\lambda_k,\lambda_l)$ and $\{,\}$ denotes the Poisson bracket. We have introduced the exterior sources J_α and K^α with respect to the φ^α and π_α, respectively, which does not alter the calculation of Green's functions. For the sake of simplicity, let us denote $\phi=(\varphi^\alpha,\lambda_m,C^\alpha,\overline{C}^\alpha)$, $\pi=(\pi_\alpha)$, $J=(J_\alpha)$, $K=(K_\alpha)$ and $I^{\mathrm{p}}=I^{\mathrm{p}}_{\mathrm{eff}}$. Thus, expression (1) can be written as

$$Z[J,K]=\int \mathcal{D}\phi\mathcal{D}\pi\exp\left\{\mathrm{i}\left[I^{\mathrm{p}}+\int \mathrm{d}^4x\,(J\phi+K\pi)\right]\right\}. \tag{4}$$

For a system with a regular (non-singular) Lagrangian, the effective canonical action is given by (3).

Let us consider a global symmetry transformation in extended phase space whose infinitesimal transformation is given by

$$\begin{cases} x^{\mu'}=x^\mu+\Delta x^\mu=x^\mu+\varepsilon_\sigma\mathcal{T}^{\mu\sigma}(x,\phi,\pi)\\ \phi'(x')=\phi(x)+\Delta\phi(x)=\phi(x)+\varepsilon_\sigma S^\sigma(x,\phi,\pi)\\ \pi'(x')=\pi(x)+\Delta\pi(x)=\pi(x)+\varepsilon_\sigma T^\sigma(x,\phi,\pi)\end{cases} \tag{5}$$

where ε_σ ($\sigma=1,2,\ldots,r$) are infinitesimal arbitrary parameters and $\mathcal{T}^{\mu\sigma}$, S^σ and T^σ are some functions of x, $\phi(x)$ and $\pi(x)$. The conformal and internal transformations are special cases of the transformation (5). Under the transformations (5), the variation of (effective) canonical action is given by (Li 1993a)

$$\begin{aligned}\delta I^{\mathrm{p}}=\int \mathrm{d}^4x\,&\bigg\{\frac{\delta I^{\mathrm{p}}}{\delta\phi}(\Delta\phi-\phi_{,\mu}\Delta x^\mu)+\frac{\delta I^{\mathrm{p}}}{\delta\pi}(\Delta\pi-\pi_{,\mu}\Delta x^\mu)\\ &+\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_{\mathrm{c}})\Delta x^\mu]+\mathrm{D}[\pi(\Delta\pi-\phi_{,\mu}\Delta x^\mu)]\bigg\}\\ =\int \mathrm{d}^4x\,\varepsilon_\sigma&\bigg\{\frac{\delta I^{\mathrm{p}}}{\delta\phi}(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})+\frac{\delta I^{\mathrm{p}}}{\delta\pi}(T^\sigma-\pi_{,\mu}\mathcal{T}^{\mu\sigma})\\ &+\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_{\mathrm{c}})\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\bigg\}\end{aligned} \tag{6}$$

where $\mathrm{D}=\mathrm{d}/\mathrm{d}t$, and

$$\frac{\delta I^{\mathrm{p}}}{\delta\phi}=-\dot{\pi}-\frac{\delta H_{\mathrm{c}}}{\delta\phi}\qquad \frac{\delta I^{\mathrm{p}}}{\delta\pi}=\dot{\phi}-\frac{\delta H_{\mathrm{c}}}{\delta\pi}. \tag{7}$$

It is supposed that the Jacobian of the transformation is equal to unity. The phase-space generating functional (4) is invariant under the transformation (5). Thus, we have

$$\begin{aligned}Z[J,K]=\int \mathcal{D}\phi\mathcal{D}\pi\exp\bigg\{&\mathrm{i}(I^{\mathrm{p}}+\delta I^{\mathrm{p}})+\mathrm{i}\int \mathrm{d}^4x\,(J\phi+K\pi+\varepsilon_\sigma J(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})\\ &+\varepsilon_\sigma K(T^\sigma-\pi_{,\mu}\mathcal{T}^{\mu\sigma})+\varepsilon_\sigma\partial_\mu[(J\phi+K\pi)\mathcal{T}^{\mu\sigma}])\bigg\}.\end{aligned} \tag{8}$$

If the (effective) canonical action is invariant under the transformation (5), $\delta I^{\mathrm{p}}=0$, and from (8) one obtains

$$
\begin{aligned}
Z[J,K] &= \int \mathcal{D}\phi\mathcal{D}\pi \exp\left\{\mathrm{i}\left[I^{\mathrm{p}}+\int \mathrm{d}^4x\,(J\phi+K\pi)\right]\right\}\bigg(1+\mathrm{i}\varepsilon_\sigma\int \mathrm{d}^4x\,\{J(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma}) \\
&\quad +K(T^\sigma-\pi_{,\mu}\mathcal{T}^{\mu\sigma})+\partial_\mu[(J\phi+K\pi)\mathcal{T}^{\mu\sigma}]\}\bigg) \\
&= \left(1+\mathrm{i}\varepsilon_\sigma\int \mathrm{d}^4x\left\{J\left(S^\sigma-\mathcal{T}^{\mu\sigma}\partial_\mu\frac{\delta}{\delta J}\right)+K\left(T^\sigma-\mathcal{T}^{\mu\sigma}\partial_\mu\frac{\delta}{\delta K}\right)\right.\right. \\
&\quad \left.\left.+\partial_\mu\left[\mathcal{T}^{\mu\sigma}\left(J\frac{\delta}{\delta J}+K\frac{\delta}{\delta K}\right)\right]\right\}\right)\bigg|_{\phi\to\frac{\delta}{\mathrm{i}\delta J},\pi\to\frac{\delta}{\mathrm{i}\delta\pi}} Z[J,K].
\end{aligned}
\tag{9}
$$

Consequently, we obtain the following results. If the (effective) canonical action is invariant under the transformation (5) in the extended phase, then, the phase-space generating functional of the Green's function satisfies the equation

$$
\begin{aligned}
&\int \mathrm{d}^4x\left\{J\left(S^\sigma-\mathcal{T}^{\mu\sigma}\partial_\mu\frac{\delta}{\delta J}\right)+K\left(T^\sigma-\mathcal{T}^{\mu\sigma}\partial_\mu\frac{\delta}{\delta K}\right)\right. \\
&\qquad \left.+\partial_\mu\left[\mathcal{T}^{\mu\sigma}\left(J\frac{\delta}{\delta J}+K\frac{\delta}{\delta K}\right)\right]\right\}\bigg|_{\phi\to\frac{\delta}{\mathrm{i}\delta J},\pi\to\frac{\delta}{\mathrm{i}\delta\pi}} Z[J,K]=0.
\end{aligned}
\tag{10}
$$

Expression (10) can be called the canonical Ward identities under the global symmetry transformation in phase space.

For the internal symmetry transformation $\mathcal{T}^{\mu\sigma}=0$; in this case expression (10) can be written as

$$
\int \mathrm{d}^4x\left[J(x)S^\sigma\left(x,\frac{\delta}{\mathrm{i}\delta J},\frac{\delta}{\mathrm{i}\delta K}\right)+K(x)T^\sigma\left(x,\frac{\delta}{\mathrm{i}\delta J},\frac{\delta}{\mathrm{i}\delta K}\right)\right]Z[J,K]=0. \tag{11}
$$

3. Conserved charge in the quantum case

The connection between the canonical continuous symmetry and conservation laws is usually referred to as the canonical Noether's theorem in classical theories (Li 1993a). In this paper the realization of a canonical symmetry (especially global symmetry) for a constrained Hamiltonian system at the quantum level is studied.

Let us start from a classical canonical action, invariant under the transformation (5). Then we quantize the system. This means that not only is the classical trajectory allowed, but also all of the possible paths, each one with its own weight. We take the classical canonical action and insert it into the phase-space generating functional (4). In the functional formalism we may make use of the local transformation in extended phase space:

$$
\begin{cases}
x^{\mu'}=x^\mu+\Delta x^\mu=x^\mu+\varepsilon_\sigma(x)\mathcal{T}^{\mu\sigma}(x,\phi,\pi) \\
\phi'(x')=\phi(x)+\Delta\phi(x)=\phi(x)+\varepsilon_\sigma(x)S^\sigma(x,\phi,\pi) \\
\pi'(x')=\pi(x)+\Delta\pi(x)=\pi(x)+\varepsilon_\sigma(x)T^\sigma(x,\phi,\pi)
\end{cases}
\tag{12}
$$

where $\varepsilon_\sigma(x)$ ($\sigma=1,2,\ldots,r$) are infinitesimal arbitrary functions and they will vanish on the boundary of the time-space domain. Under the transformation (12) the variation of canonical action is given by

$$
\delta I^{\mathrm{p}}=\int \mathrm{d}^4x\,\varepsilon_\sigma(x)\left\{\frac{\delta I^{\mathrm{p}}}{\delta\phi}(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})+\frac{\delta I^{\mathrm{p}}}{\delta\pi}(T^\sigma-\pi_{,\mu}\mathcal{T}^{\mu\sigma})\right.
$$

$$+\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\Big\}$$

$$+\int \mathrm{d}^4x\,\{[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]\partial_\mu\varepsilon_\sigma(x)+\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})\mathrm{D}\varepsilon_\sigma(x)\}. \tag{13}$$

Because the canonical action is invariant under the global transformation (5), then the first integral in expression (13) is equal to zero. According to the boundary condition of $\varepsilon_\sigma(x)$, expression (13) can be written as

$$\begin{aligned}\delta I^{\mathrm{p}} &= \int \mathrm{d}^4x\,\{[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]\partial_\mu\varepsilon_\sigma(x)+\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})\mathrm{D}\varepsilon_\sigma(x)\}\\ &= -\int \mathrm{d}^4x\,\varepsilon_\sigma(x)\{\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\}.\end{aligned} \tag{14}$$

The phase-space generating functional (4) is invariant under the local transformation (12) and, hence, leads to

$$\begin{aligned}Z[J,K] = \int \mathcal{D}\phi\mathcal{D}\pi\exp\left\{\mathrm{i}\int \mathrm{d}^4x\,[\pi\dot{\phi}-\mathcal{H}_c+J\phi+K\pi]\right\}\\ \times\Big(1-\mathrm{i}\int \mathrm{d}^4x\,\varepsilon_\sigma(x)\{\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\\ -J(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})-K(T^\sigma-\pi_{,\mu}\mathcal{T}^{\mu\sigma})\}\\ +\mathrm{i}\int \mathrm{d}^4x\,\partial_\mu[(J\phi+K\pi)\varepsilon_\sigma(x)\mathcal{T}^{\mu\sigma}]\Big).\end{aligned} \tag{15}$$

We functionally differentiate (15) with respect to $\varepsilon_\sigma(x)$, and set all exterior sources equal to zero, i.e. $J=K=0$, and obtain

$$\int \mathcal{D}\phi\mathcal{D}\pi\{\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\}\exp\left\{\mathrm{i}\int \mathrm{d}^4x\,(\pi\dot{\phi}-\mathcal{H}_c)\right\}=0$$
$$(\sigma=1,2,\ldots,r). \tag{16}$$

It follows that

$$\langle 0|T^*\{\partial_\mu[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]+\mathrm{D}[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]\}|0\rangle=0 \qquad (\sigma=1,2,\ldots,r). \tag{17}$$

The symbol T^* stands for the covariantized T product (Surra and Young 1973). Expression (17) can also be written as

$$\partial_\mu\langle 0|T[(\pi\dot{\phi}-\mathcal{H}_c)\mathcal{T}^{\mu\sigma}]|0\rangle+\mathrm{D}\langle 0|T[\pi(S^\sigma-\phi_{,\mu}\mathcal{T}^{\mu\sigma})]|0\rangle=0 \qquad (\sigma=1,2,\ldots,r). \tag{18}$$

We now take a cylinder in four-dimensional space, the axis of which is directed along the t axis and the upper and lower bottom V_1 and V_2 are two like-space hypersurfaces $t=t_1$ and $t=t_2$ respectively. If we assume that the fields approach zero rapidly enough, then taking the integral of (18) on this cylinder, from Gauss's theorem of four-dimensional space we can neglect the contribution to the boundary of the infinite cylinder connecting V_1 and V_2. Thus, we have

$$\int_V \mathrm{d}^3x\,\langle 0|T[\pi S^\sigma-\phi_{,k}\tau^{k\sigma}-\mathcal{H}_c\mathcal{T}^{\mu\sigma}]|0\rangle=\text{constant} \qquad (\sigma=1,2,\ldots,r). \tag{19}$$

Equation (19) expresses the conserved charge at the quantum level connected with the classical canonical Noether theorem in phase space (Li 1993b), which means that the integral in three-dimensional space of the expectation values of some operators on the ground state is equal to a constant. These results hold true for the anomalies-free theories.

For a system with a singular Lagrangian, (19) holds true on the constrained hypersurface if the canonical action and constraint conditions are invariant under the transformation (12).

The advantage of the above derivation for conserved charge is that one need not simplify by carrying out explicit integrations over momenta in the phase-space generating functional. In the general case this cannot be done, and the phase-space generating functional cannot be represented in the so-called Lagrangian form, i.e. in the form of a functional integral only over the 'coordinates' of the expression containing a certain effective Lagrangian in configuration space.

4. Local transformation and generalized canonical Ward identities

Let us now consider a functional integral

$$Z_F[J,K]=\int \mathcal{D}\phi\mathcal{D}\pi\, F(\phi,\pi)\exp\left\{\mathrm{i}\left[I^{\mathrm{P}}+\int \mathrm{d}^4x\,(J\phi+K\pi)\right]\right\} \tag{20}$$

where $F(\phi,\pi)$ is a functional of the variables ϕ and π. For the case $J=K=0$, the expectation value of the operator $\hat{F}$ on the ground state is given by equation (20). If one takes the product of the fields $\phi(x)$, $F=\phi(x_1)\phi(x_2)\cdots\phi(x_n)$ for F, one can obtain the Green's function from (20).

Consider an infinitesimal local transformation in an extended phase space:

$$\begin{cases} x^{\mu'}=x^\mu+\Delta x^\mu=x^\mu+R^\mu_\sigma\varepsilon^\sigma(x)\\ \phi'(x')=\phi(x)+\Delta\phi(x)=\phi(x)+S_\sigma\varepsilon^\sigma(x)\\ \pi(x')=\pi(x)+\Delta\pi(x)=\pi(x)+T_\sigma\varepsilon^\sigma(x)\end{cases} \tag{21}$$

where $\varepsilon^\sigma(x)$ $(\sigma=1,2,\ldots,r)$ are infinitesimal arbitrary functions, the values of $\varepsilon^\sigma(x)$ and their derivatives on the boundary of the time-space domain vanish, and

$$\begin{aligned} &R^\mu_\sigma=A^{\mu\nu(k)}_\sigma\partial_{\nu(k)} \qquad S_\sigma=B^{\nu(l)}_\sigma\partial_{\nu(l)} \qquad T_\sigma=C^{\nu(m)}_\sigma\partial_{\nu(m)}\\ &\nu(n)=\underbrace{\nu\mu\cdots\lambda\rho}_{n} \qquad \partial_{\nu(n)}=\partial_\nu\partial_\mu\cdots\partial_\lambda\partial_\rho \end{aligned} \tag{22}$$

where A, B and C are functions of x, ϕ and π. The Jacobian of transformation (21) is denoted by $J[\phi,\pi,\varepsilon]$. Owing to the boundary conditions of $\varepsilon^\sigma(x)$, the functional integral (20) is invariant under the transformation (21), which implies

$$\left.\frac{\delta Z_F}{\delta\varepsilon^\sigma}\right|_{\varepsilon^\sigma=0}=0.$$

From (6) and (20) one obtains the generalized canonical identities in phase space:

$$\begin{aligned}\Bigg[J^0_\sigma&+\tilde{S}_\sigma\left(\frac{\delta F}{\delta\phi}\right)-\tilde{R}^\mu_\sigma\left(\phi_{,\mu}\frac{\delta F}{\delta\phi}\right)+\tilde{T}_\sigma\left(\frac{\delta F}{\delta\pi}\right)-\tilde{R}^\mu_\sigma\left(\pi_{,\mu}\frac{\delta F}{\delta\pi}\right)+\mathrm{i}\tilde{S}_\sigma\left(F\frac{\delta I^{\mathrm{P}}}{\delta\phi}\right)\\ &-\mathrm{i}\tilde{R}^\mu_\sigma\left(\phi_{,\mu}F\frac{\delta I^{\mathrm{P}}}{\delta\phi}\right)+\mathrm{i}\tilde{T}_\sigma\left(F\frac{\delta I^{\mathrm{P}}}{\delta\pi}\right)-\mathrm{i}\tilde{R}^\mu_\sigma\left(\pi_{,\mu}F\frac{\delta I^{\mathrm{P}}}{\delta\pi}\right)+\mathrm{i}\tilde{S}_\sigma(FJ)\\ &-\mathrm{i}\tilde{R}^\mu_\sigma(\phi_{,\mu}FJ)+\mathrm{i}\tilde{T}_\sigma(FK)-\mathrm{i}\tilde{R}^\mu_\sigma(\pi_{,\mu}FK)\Bigg]\Bigg|_{\phi\to\frac{\delta}{\mathrm{i}\delta J},\pi\to\frac{\delta}{\mathrm{i}\delta K}}Z_F[J,K]=0\end{aligned} \tag{23}$$

where

$$J^0_\sigma=\left.\frac{\delta J[\phi,\pi,\varepsilon]}{\delta\varepsilon^\sigma}\right|_{\varepsilon^\sigma=0}.$$

In the above expressions we have used $J[\phi, \pi, 0] = 1$. The $\tilde{R}^{\mu}_{\sigma}$, $\tilde{S}_{\sigma}$ and $\tilde{T}_{\sigma}$ are adjoint operators with respect to R^{μ}_{σ}, S_{σ} and T_{σ} respectively (Li 1987). For the case $F = 1$, the generalized canonical Ward identities (23) converts to the Ward identities in phase space (Li 1994a).

5. An application

As an application of the previous results, we study a system of interacting polaron and photon. The polaron is basic to the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity of metals. A Lagrangian which describes the electron–photon (polaron) interaction was proposed in $1+1$-dimensional space by Rodriguez-Nunez (1990). Now a case involving an electromagnetic field will be discussed. The inclusion of an electromagnetic field can be done by requiring that $\partial_{\mu} \to \partial_{\mu} - \mathrm{i}eA_{\mu}$. Also, we can add the electromagnetic invariant $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ are the usual electromagnetic tensors $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and $A_{\mu}(x)$ represents an electromagnetic field. The Lagrangian for a system of interacting polaron and photon in $1+3$-dimensional space can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \psi^*\left[\mathrm{i}(\partial_0 - \mathrm{i}eA_0) + \frac{1}{2m}(\nabla - \mathrm{i}e\vec{A})^2\right]\psi + \frac{1}{2}[\rho(\partial_0 q)^2 - s(\partial_i q)^2] - Gq\psi^*\psi \tag{24}$$

where $\psi(x)$ represents the electronic field, $q(x)$ represents the phononic field, $A_{\mu}(x) = (A_0(x), \vec{A}(x))$, and ρ, s and G are parameters (Rodriguez-Nunez 1990).

The canonical momenta π_{ψ}, π_{ψ^*}, π_q and π^{μ} conjugate to ψ, ψ^*, q and A_{μ} are

$$\begin{aligned} \pi_{\psi} &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \mathrm{i}\psi^* & \qquad \pi_{\psi^*} &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0 \\ \pi_q &= \frac{\partial\mathcal{L}}{\partial\dot{q}} = \rho\dot{q} & \qquad \pi^{\mu} &= \frac{\partial\mathcal{L}}{\partial\dot{A}_{\mu}} = -F^{0\mu} \end{aligned} \tag{25}$$

respectively. The canonical Hamiltonian density of the system is given by

$$\begin{aligned} \mathcal{H}_c &= \pi_{\psi}\dot{\psi} + \pi_{\psi^*}\dot{\psi}^* + \pi_q\dot{q} + \pi^{\mu}\dot{A}_{\mu} - \mathcal{L} \\ &= \frac{1}{2}\pi_i^2 - A^0\partial_i\pi_i + \frac{1}{4}F_{ik}F_{ik} - e\psi^*A_0\psi - \frac{1}{2m}\psi^*[(\nabla - \mathrm{i}e\vec{A})^2]\psi \\ &\quad + \frac{1}{2\rho}\pi_q^2 + \frac{s}{2}(\nabla q)^2 + Gq\psi^*\psi. \end{aligned} \tag{26}$$

The primary constraints are

$$\theta_1 = \pi_{\psi} - \mathrm{i}\psi^* \approx 0 \tag{27}$$

$$\theta_2 = \pi_{\psi^*} \approx 0 \tag{28}$$

$$\Lambda_1 = \pi_0 \approx 0. \tag{29}$$

The total Hamiltonian is given by

$$H_{\mathrm{T}} = \int \mathrm{d}^3x(\mathcal{H}_c + \lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\Lambda_1) \tag{30}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$ are Lagrangian multipliers. The stationary condition of the primary constraints θ_i $(i = 1, 2)$, $\{\theta_i, H_{\mathrm{T}}\} \approx 0$, yield the equations

$$\mathrm{i}\lambda_2 \approx e\psi^*A_0 + \frac{1}{2m}[\nabla^2\psi^* + \mathrm{i}e\nabla\psi^*\cdot\vec{A} + \mathrm{i}e\nabla\cdot(\psi^*\vec{A}) - e^2A^2\psi^*] - Gq\psi^* \tag{31}$$

$$\mathrm{i}\lambda_1 \approx -\mathrm{e}\psi A_0 + \frac{1}{2m}[\nabla^2\psi - \mathrm{ie}\nabla\cdot(\psi\vec{A}) - \mathrm{ie}\nabla\psi\cdot\vec{A} - \mathrm{e}^2 A\psi] + Gq\psi. \tag{32}$$

Equations (31) and (32) fix the Lagrangian multipliers, but do not produce additional constraints. The stationary condition of the primary constraint Λ_1, $\{\Lambda_1, H_{\mathrm{T}}\} \approx 0$, yields the secondary constraint

$$\chi = \partial_i \pi_i + \mathrm{e}\psi^*\psi \approx 0. \tag{33}$$

The stationary condition of this secondary constraint does not produce another constraint. From the linear combination of the constraints θ_1, θ_2 and χ, one sets

$$\Lambda_2 = \chi - \mathrm{ie}(\psi\theta_1 - \psi^*\theta_2) = \partial_i\pi_i - \mathrm{ie}(\psi\pi_\psi - \psi^*\pi_{\psi^*}) \approx 0. \tag{34}$$

It is easy to check that Λ_1 and Λ_2 are first-class constraints and that θ_1 and θ_2 are second-class constraints.

We adopt the quantization of path integrals. For each first-class constraint a gauge condition should be chosen. Consider the Coulomb gauge

$$\Omega_2 = \partial_i A_i \approx 0. \tag{35}$$

For consistency of Ω_2, this has to be $\partial_0\Omega_2 \approx 0$, and we may take another gauge condition

$$\Omega_1 = \partial_i\pi_i + \partial_i\partial_i A^0 \approx 0. \tag{36}$$

The factors $\det|\{\Omega_\alpha, \Lambda_\beta\}|$ and $\det|\{\theta_i, \theta_j\}|$ are independent of field variables. Thus, we can omit this factor from the generating functional ($1a$). Hence, the phase-space generating functional for a system of interacting polaron and photon is given by

$$\begin{aligned} Z[J,K] &= Z[J^\mu, K_\mu, \xi, K_1, \zeta, K_2, \eta, K_3, V, X, Y] \\ &= \int \mathcal{D}A_\mu \mathcal{D}\pi^\mu \mathcal{D}\psi \mathcal{D}\pi_\psi \mathcal{D}\psi^* \mathcal{D}\pi_{\psi^*} \mathcal{D}q \mathcal{D}\mu \mathcal{D}\omega \mathcal{D}\nu \\ &\quad \times \exp\left\{\mathrm{i}\int \mathrm{d}^4x\, [\mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} + J^\mu A_\mu + K^\mu \pi_\mu + \xi\psi + K_1\pi_\psi + \zeta\psi^* + K_2\pi_{\psi^*} \right. \\ &\quad \left. + \eta q + K_3\pi_q + V_k\mu_k + X_l\omega_l + Y_i\nu_i]\right\} \\ &= \int \mathcal{D}\phi\mathcal{D}\pi \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} + J\phi + K\pi]\right\} \end{aligned} \tag{37}$$

where

$$\begin{aligned} &\phi = (A_\mu, \psi, \psi^*, q, \mu, \omega, \nu) \qquad \pi = (\pi_\mu, \pi_\psi, \pi_{\psi^*}, \pi_q) \\ &J = (J_\mu, \xi, \zeta, \eta, V_k, X_l, Y_i) \qquad K = (K^\mu, K_1, K_2, K_3) \\ &\mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} = \pi_\psi\dot{\psi} + \pi_{\psi^*}\dot{\psi}^* + \pi_q\dot{q} + \pi^\mu\dot{A}_\mu - \mathcal{H}_{\mathrm{c}} + \mu_k\Lambda_k + \omega_l\Omega_l + \nu_i\theta_i \end{aligned} \tag{38}$$

and $\mu_k = \mu_k(x)$, $\omega_l = \omega_l(x)$ and $\nu_i = \nu_i(x)$ are multiplier fields.

The (effective) canonical action is invariant under the global gauge transformation

$$\psi'(x) = \mathrm{e}^{-\mathrm{ie}\varepsilon}\psi(x) \qquad \pi'_\psi(x) = \mathrm{e}^{\mathrm{ie}\varepsilon}\pi_\psi(x). \tag{39}$$

From (19) we have the conservation law of charge

$$Q = \mathrm{e}\int_V \mathrm{d}^3x\,\langle 0|\psi^*\psi|0\rangle = \text{constant}. \tag{40}$$

Similarly, from the translation invariance of time-space, one can obtain the conservation law of energy at the quantum level.

Now let us consider the local transformation in phase space:

$$\begin{cases} A'_\mu(x) = A_\mu(x) + \partial_\mu \varepsilon(x) & \pi^{\mu'}(x) = \pi^\mu(x) \\ \psi'(x) = \exp(-\mathrm{i}\varepsilon(x))\psi(x) & \pi'_\psi(x) = \exp(\mathrm{i}\varepsilon(x))\pi_\psi(x) \\ \psi^{*'}(x) = \exp(\mathrm{i}\varepsilon(x))\psi^*(x) & \pi'_{\psi^*}(x) = \pi_{\psi^*}(x). \end{cases} \tag{41}$$

The Jacobian of the transformation (41) is equal to unity. The generating functional (37) is invariant under the transformation (41).

Let us set $F(\phi, \pi) = 1$ in (23). Thus, in this case the generalized canonical Ward identities (23) become

$$\int \mathcal{D}\phi\mathcal{D}\pi\,[\nabla^2\omega_2 - \partial_\mu J^\mu - \mathrm{i}\xi\psi + \mathrm{i}\zeta\psi^* + \mathrm{i}K\pi_\psi]\exp\left\{\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}^{\mathrm{p}}_{\mathrm{eff}} + J\phi + K\pi]\right\} = 0 \tag{42}$$

or

$$\left[\nabla^2\frac{\delta}{\delta x_2} - \partial_\mu J^\mu - \xi\frac{\delta}{\delta\xi} + \zeta\frac{\delta}{\delta\zeta} + K\frac{\delta}{\delta K}\right]Z[J, K] = 0. \tag{43}$$

Let $Z[J, K] = \exp\{\mathrm{i}W[J, K]\}$; thus, expression (43) can be written as

$$\left[\nabla^2\frac{\delta}{\delta x_2} - \partial_\mu J^\mu W^{-1} - \xi\frac{\delta}{\delta\xi} + \zeta\frac{\delta}{\delta\zeta} + K\frac{\delta}{\delta K}\right]W[J, K] = 0. \tag{44}$$

Using a functional Legendre transformation (Li 1994a), one obtains

$$\Gamma[\phi, \pi] = W[J, K] - \int \mathrm{d}^4x\,(J\phi + K\pi) \tag{45}$$

$$\frac{\delta W}{\delta J(x)} = \phi(x) \qquad \frac{\delta\Gamma}{\delta\phi(x)} = -J(x) \tag{46a}$$

$$\frac{\delta W}{\delta K(x)} = \pi(x) \qquad \frac{\delta\Gamma}{\delta\pi(x)} = -K(x). \tag{46b}$$

Hence, expression (44) can be written as

$$\nabla^2\omega_2(x) + \partial^\mu\left(\frac{\delta\Gamma}{\delta A^\mu(x)}\right) + \psi(x)\frac{\delta\Gamma}{\delta\psi(x)} - \psi^*(x)\frac{\delta\Gamma}{\delta\psi^*(x)} + \pi_\psi\frac{\delta\Gamma}{\delta\pi_\psi(x)} = 0. \tag{47}$$

We functionally differentiate (47) with respect to $\psi^*(x_1)$ and $\psi(x_2)$ respectively, and set all fields equal to zero, i.e. $\psi = \psi^* = \pi_\psi = A_\mu = \pi^\mu = q = \pi_q = \mu = \omega = \nu = 0$, to obtain

$$\partial^\mu_{x_1}\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_3)\delta\psi(x_2)\delta A^\mu(x_1)} = \delta(x_1 - x_2)\frac{\delta^2\Gamma[0]}{\delta\psi(x_1)\delta\psi^*(x_3)} - \delta(x_1 - x_3)\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(x_2)}. \tag{48}$$

Expression (48) indicates that there are some relationships among the propagators and proper vertices. Differentiating equation (47) many times with respect to the field variables, one can obtain various Ward identities for proper vertices in phase space.

If one takes $\varepsilon(x) = \varepsilon_0 q(x)$ in equation (41), where ε_0 is an infinitesimal parameter, then this transformation is a special case of the global transformation (5). The canonical Ward identities (11) for a global transformation in this case can be written as

$$\int \mathrm{d}^4x\left[J^\mu\partial_\mu - \mathrm{i}\xi\frac{\delta}{\delta\xi} + \mathrm{i}\zeta\frac{\delta}{\delta\zeta} + \mathrm{i}K_1\frac{\delta}{\delta K_1}\right]\frac{\delta}{\delta\eta}Z[J^\mu, K_\mu, \xi, K_1, \zeta, K_2, \eta, K_3, V, X, Y] = 0. \tag{49}$$

Since the Hamiltonian is quadratic in canonical momenta, the integration over the canonical momenta in (37) can be performed and the effective Lagrangian in configuration space can be obtained which contains a gauge-fixing term in the Coulomb gauge. The Ward identity (48) can also be derived by using a configuration-space generating functional (Li and Xie 1995).

6. Conclusion and discussion

The canonical symmetry for a system with a singular Lagrangian at the quantum level is studied. The canonical Ward identities under the global and local transformation have been deduced. The canonical Noether theorem at the quantum level is also obtained. For a constrained Hamiltonian system the effective canonical action is different from the classical action. The existence of conserved quantities is required so that the classical canonical action is symmetric and the constraint conditions are preserved in the constrained hypersurface under the corresponding transformation. To illustrate the above considerations, a Lagrangian of interacting polaron and photon in $1+3$-dimensional space is proposed. The conserved charges and Ward identities for proper vertices have been obtained.

The above formulation is based on the path-integral quantization formalism following the Faddeev (1970) and Senjanovic (1976) procedure, in which we have introduced the exterior sources for canonical momenta. The most general, different quantization scheme is that proposed by Batalin and Vilkovisky (1977) and Fradkin and Vilkovisky (1975) (BFV procedure), who developed a canonical approach to Becchi–Rouet–Stora (BRS) or Becchi–Rouet–Stora–Tyutin (BRST) quantization (Henneaux 1992). According to the BFV procedure, one introduces the Lagrangian multiplier fields for first-class constraints along with its corresponding canonical momenta, and introduce the Fermionic ghost fields together with their respective canonical momenta. In the enlarged phase space all the canonical variables are treated as dynamical variables. The functional integral can be formed once the conserved BRST charges have been found and the Fermi function has been chosen. For the case of interest, the BFV approach is equivalent to a simpler approach based on the construction of the BRS (or BRST) Hamiltonian associated with the effective Lagrangian obtained after fixing the gauge in the configuration-space generating functional, and using the Faddeev–Popov trick through a transformation of the generating functional in this gauge (Abdalla *et al* 1991). It has been illustrated by concrete examples that, after integrating over all canonical momenta, the expression for the phase-space generating functional of Green's function for a constrained Hamiltonian system which has been obtained by the canonical quantization given by the Faddeev and Senjanovic procedure, can also be written in the form of a configuration-space generating functional containing a certain effective Lagrangian (Sundermeyer 1982, Gitman *et al* 1990). Thus, one could verify that our formulation for the case when the integration over canonical momenta in the generating functional can be carried out is consistent and equivalent to the formulation obtained through canonical BRST quantization of an interesting system (Li 1994a–c). The advantage of our derivation is that one does not need to carry out the integration over the momenta in the phase-space generating functional of Green's function as usually necessary; in the general case this integration cannot be carried out.

The application of our results to Yang–Mills theory can proceed in the same way as discussed in section 5. The effective canonical action and the constraint conditions are invariant under scale transformation for ghost fields (Li 1994a):

$$C^a(x) \rightarrow e^{\theta} C^a(x) \qquad \overline{C}^a(x) \rightarrow e^{-\theta}\overline{C}^a(x) \tag{50}$$

where θ is a constant even parameter. The conserved ghost charge at the quantum level can be obtained as

$$Q_c = \int_V d^3x \, \langle 0|T^*[\pi^a(\overleftrightarrow{\partial}{}^0 \delta^a_b + f^a_{eb} A^{0e})C^b]|0\rangle = \text{constant} \tag{51}$$

which coincides with the results obtained by means of the so-called effective action theories (Gitman *et al* 1990).

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